Review: Proof of Bishop Volume Comparison Theorem with Isoperimetric Technique

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Abstract: The Bishop's theorem states that a Riemannian manifold M^n with Ricci curvature larger than the standard *n*-sphere has less volume. This paper mainly reviews the Hubert Bray's proof of Bishop's volume comparison theorem with isoperimetric technique. Some details in the proof are optimized to address this issue more clearly.

1. Introduction

It is well known that curvature conditions have a significant impact on manifold's geometry. A classic volume comparison theorem in Riemannian geometry is the following Bishop's theorem.

Theorem1.1(Bishop-Gromov) *If $\left(M^n, g\right)$ is complete Riemannian manifold with $Ric \geq \left(m-1\right)k$, and $p \in M$ is an arbitrary point, then the function $r \mapsto \dfrac{\mathrm{Vol}\left(B_r\left(p\right)\right)}{\mathrm{Vol}\left(B_r^k\right)}$ is a non-increasing function which tends to 1 as r goes to 0, where $B_r\left(p\right)$ is the geodesic ball in M^n centered at p, B_r^k is a geodesic ball of radius r in the space form M_k^n. In particular, $\mathrm{Vol}\left(B_r\left(p\right)\right) \leq \mathrm{Vol}\left(B_r^k\right)$.*

The Bishop's volume comparison theorem has been studied since it was first demonstrated by Bishop in 1963 [1]. Literature[8] provides a proof utilizing the geodesic ball and index form. In the area of differential geometry, this theorem is widely used. It can be used, for instance, to provide an elementary proof of the maximal diameter theorem [2], namely for a complete Riemannian *n*-manifold M^n whose Ricci curvature is bounded below by 1, if the diameter of M is equal to π, then M is isometric to $S^n\left(1\right)$. A detailed proof of this theorem using Bishop's theorem can be found in [3].

In this paper, we mainly review Bray's proof of a special case of Bishop's theorem[4].

Theorem1.2(Bishop) *Let (S^n, g_0) be the standard metric (with any scaling) on S^n with constant Ricci curvature $Ric_0 \cdot g_0$. If $\left(M^n, g\right)$ is a complete Riemannian manifold $\left(n \geq 2\right)$ with $Ric\left(g\right) \geq Ric_0 \cdot g$, then $Vol(M^n) \leq Vol(S^n)$.*

2. Preparations

In this section, We discuss the definition of volume and derive the inequalities needed to prove the Bishop's theorem.

The definition of the volume of M in Theorem 1.2 is ambiguous. For simplicity, in the following discussion we may just consider the orientable case. There are two ways to define the volume in the non-orientable case. One way is to use the following definition, and the other is to define the non-orientable volume of the manifold as half of the volume of its orientable double cover. While in geometric measure theory, the volume of the manifold is defined as its Hausdorff measure.

Since the manifolds we are discussing about are complete, the exponential map plays an important role in the definition of volume. Recall that on an open chart, the volume form of a

Riemannian manifold $\left(M^n, g\right)$ is

$$d\mathrm{Vol} = \sqrt{G \circ x^{-1}}\, dx_1 dx_2 \ldots dx_n.$$

where $G = \det\left(g_{ij}\right)$ and $dx_1 \ldots dx_n$ is the Lebesgue measure on $\mathbb{R}^n$.

It is known that for a connected complete Riemannian manifold $\left(M^n, g\right)$, $\exp_p$ is a diffeomorphism onto the dense open set $M \setminus Cut(p)$. If we denote $\Sigma(p) = \exp_p^{-1}\left(M \setminus Cut(p)\right)$, $\{exp_p^{-1}, M \setminus Cut(p), \Sigma(p)\}$ is an open coordinate chart on M (we identify the T_pM with $\mathbb{R}^n$). Here we define the function

$$\mu(r,\Theta) = \sqrt{G \circ exp_p(r,\Theta)}\, r^{n-1} \text{ if } r\Theta \in \Sigma(p)$$

and if we suppose $d\Theta$ is is the usual surface measure on S^{n-1}, then on $\Sigma(p)$ we have

$$d\mathrm{Vol} = \mu(r,\Theta)\, dr d\Theta.$$

And we define $\mu(r,\Theta) = 0$ if $r\Theta \in T_pM \setminus \Sigma(p)$.This is because firstly $Cut(p)$ is a set of zero measure, which has no affect on the result of the integration. Secondly the images of points outside $\Sigma(p)$ under the exponential map are in $Cut(p)$, or have been mapped by some points in $\Sigma(p)$. Note that by definition, the geodesic ball

$\overline{B_r(p)} = \exp_p\left(\overline{B_r(0)}\right) = \exp_p\left(\overline{B_r(0)} \cap \overline{\Sigma(p)}\right)$ Where $B_r(0)$ is the ball in T_pM.

Since $Cut(p)$ is of measure zero in M, the volume of the geodesic ball is

$$\mathrm{Vol}\left(B_r(p)\right) = \int_{B_r(0) \cap \Sigma(p)} \mu(r,\Theta)\, dr d\Theta = \int_{B_r(0)} \mu(r,\Theta)\, dr d\Theta.$$

And we define

$\mathrm{Vol}(M) = \mathrm{Vol}\left(B_d(p)\right)$, where d is the diameter of M.

Using the definition above, we can directly prove the above theorem 1.2 with theorem 1.1. According to the Bonnet-Myers theorem, M 's diameter is less than π. To arrive at the conclusion of Theorem 1.2, we simply consider r equal to π in Theorem 1.1.

Since we are dealing with compact complete manifolds, the above definition using exponential map is a natural choice. The concept of volume given above corresponds to the definition of volume in Geometric Measure Theory on a Riemannian manifold, since at this time Riemannian measure matches Hausdorff measure.

The following information about isoperimetric hypersurface is required before we move on to the proof of this special case of Bishop's theorem.

Let Σ^{n-1}. be an orientable smooth compact surface without boundary in (M^n, g). We define a variation of Σ^{n-1} as follows. For $-\delta < t < \delta$ and $x \in \Sigma^{n-1}$, suppose $\Sigma^{n-1}(x,t)$ which takes values in M^n, is smooth, $\Sigma^{n-1}(t) = \{\Sigma^{n-1}(x,t) \mid x \in \Sigma^{n-1}\}$ is a smooth family of surfaces around Σ^{n-1}, and the vector $\frac{\partial \Sigma^{n-1}(x,t)}{\partial t}$ is perpendicular to $\Sigma^{n-1}(t)$ at $\Sigma^{n-1}(x,t)$. Let $\vec{\mu}(x,t)$ be the outward-pointing unit normal to $\Sigma^{n-1}(t)$ at $\Sigma^{n-1}(x,t)$, so that we must have

$$\frac{\partial \Sigma^{n-1}(x,t)}{\partial t} = \eta(x,t)\, \vec{\mu}(x,t) \tag{1}$$

for some real-valued function $\eta(x,t)\left(\eta\left(\Sigma^{n-1}(x,t)\right)\right)$. Let $d\mu(x)$ be the volume form on Σ^{n-1}, $\mathrm{II}(x)$ be the second fundamental form of Σ^{n-1} in M^n at x with respect to $\mu(x)$, and

$H(x) = trace(\mathrm{II}(x))$ be the mean curvature of Σ^{n-1} at x. Let $d\mu(x,t)$ be the volume form on $\Sigma^{n-1}(x,t)$, $\mathrm{II}(x,t)$ be the second fundamental form of $\Sigma^{n-1}(x,t)$ in M^n at $\Sigma^{n-1}(x,t)$ with respect to $\mu(x,t)$, and $H(x,t) = trace(\mathrm{II}(x,t))$ be the mean curvature of $\Sigma^{n-1}(t)$ at $\Sigma^{n-1}(x,t)$. Then we have

$$\frac{\partial}{\partial t} d\mu(x,t) = H(x,t)\eta(x,t)d\mu(x,t). \qquad (2)$$

$$\frac{\partial}{\partial t} H(x,t) = -\Delta_{\Sigma(t)}\eta(x,t) - \eta(x,t)\left|\mathrm{II}(x,t)\right|^2 - \eta(x,t)Ric_{M^n}(\vec{\mu}(x,t),\vec{\mu}(x,t)). \qquad (3)$$

The proof of the above euqations can be found in the appendix of [2]. From equation(3), we can get a useful fact.

Definition2.1 *Let (M^n, g) be a complete Riemannian n-manifold. Define the function* $A(V) = \inf_R \{Area(\partial R) \mid \mathrm{Vol}(R) = V\}$, *where R is any rectifiable current in M^n, Vol(R) is the n dimensional volume(the Hausdorff measure) of R, and Area(∂R) is the $n-1$ dimensional volume of ∂R. If there exists a rectifiable currents R with Vol(R) = V such that Area(∂R) $= A(V)$, then we say that $\Sigma = \partial R$ minimizes area with the given volume constraint.*

Lemma2.2 *If Σ minimizes area with the given volume constraint, Σ is smooth, then Σ has constant mean curvature.*

Remark: This property can be seen as a generalization of the minimum surface's zero mean curvature property.

Proof: We prove it by contradiction. Suppose that Σ does not have constant mean curvature. Then consider a smooth variation $\Sigma(t)$ of Σ. Because, $A(t) = \int_{\Sigma(t)} d\mu(x,t)$,

so, by (2), we get $A'(0) = \int_{\Sigma(t)} \frac{\partial}{\partial t} d\mu(x,t)|_{t=0} = \int_{\Sigma} H(x,0)\eta(x,0)d\mu(x,0)$.

On the other hand, we have $V'(0) = \int_{\Sigma} \eta(x,0)d\mu(x,0)$.

We can always find an $\eta(x,0)$ such that $A'(0) < 0$ and $V'(0) = 0$ unless $H(x,0)$ equals a constant, this leads to a contradition.

Bray's proof depends on the observation that the function $A(V)$ we defined above has exactly two zero points, namely 0 and the Vol(M). As a result, from this perspective, the Bishop's theorem is equivalent to comparing the zero points of the function $A(V)$, which transforms the original geometric question into an analytical one. So the next thing to do is to investigate the analytical properties of $A(V)$ under given curvature condition.

Since the manifolds we are dealing with have $Ric(g) \geq \delta > 0$, according to Bonnet-Myers theorem, these manifolds are compact, so there will always exist a minimizer $\Sigma(V)$ (not necessarily unique) for all V [5][6]. For simplicity, we only take into account the smooth case here because isoperimetric hypersurfaces may have singularities in dimensions greater than seven. Readers interested in the singular case can refer to [7], which demonstrates that in singular case the following equation (6) and the aforementioned lemma 2.2 in the sense of almost everywhere still hold true.

And now we do a unit normal variation on $\Sigma(V_0)$, that is, let $\Sigma_{V_0}(t)$ be the surface created by flowing $\Sigma(V_0)$ out at every point in the normal direction at unit speed for time t. Since $\Sigma(V_0)$ is smooth, we can do this variation for $t \in (-\delta, \delta)$ for some $\delta > 0$. Abusing the notation slightly, we can also parameterize these surfaces by their volumes as $\Sigma_{V_0}(V)$ so that $V=V_0$ will correspond to $t=0$. Let $A_{V_0}(V)=Area(\Sigma_{V_0}(V))$. Then $A(V_0)=A_{V_0}(V_0)$ and $A(V)\le A_{V_0}(V)$ since $\Sigma_{V_0}(V)$ is not necessarily minimizing for its volume. Hence, if $A(V)$ has second derivative at V_0, we have $A''(V_0)\le A_{V_0}''(V_0)$.

Now under the assumption that $Ric(g)\ge \epsilon Ric_0\cdot g$, we compute $A_{V_0}''(V_0)$. We follow the notation discussed above, $A_{V_0}(t)=\int_{\Sigma_{V_0}(t)} d\mu$, then by (2) $A_{V_0}'(t)=\int_{\Sigma_{V_0}(t)} Hd\mu$.

And since $V'(t)=\int_{\Sigma_{V_0}(t)} d\mu$, by chain rule, we get

$$A_{V_0}'(V_0)=\frac{A_{V_0}'(0)}{V'(0)}=H \quad \text{H is constant on } \Sigma_{V_0}(0)$$

Since

$$\frac{d^2A}{dV^2}=\frac{d}{dV}\left(\frac{dA}{dV}\right)=\frac{d}{dV}\left(\frac{\frac{dA}{dt}}{\frac{dV}{dt}}\right)=\frac{\frac{d^2A}{dt^2}\frac{dt}{dV}\frac{dV}{dt}-\frac{dA}{dV}\frac{d^2V}{dt^2}}{\left(\frac{dV}{dt}\right)^2}=\frac{\frac{d^2A}{dt^2}-\frac{dA}{dV}\frac{d^2V}{dt^2}}{\left(\frac{dV}{dt}\right)^2},$$

so we get

$$A_{V_0}''(V_0)=\frac{A_{V_0}''(0)-A_{V_0}'(V_0)V''(0)}{V'(0)^2}. \tag{4}$$

Using (2) and (3), we get

$$\frac{d^2A}{dV^2}=\frac{d}{dV}\left(\frac{dA}{dV}\right)=\frac{d}{dV}\left(\frac{\frac{dA}{dt}}{\frac{dV}{dt}}\right)=\frac{\frac{d^2A}{dt^2}\frac{dt}{dV}\frac{dV}{dt}-\frac{dA}{dV}\frac{d^2V}{dt^2}}{\left(\frac{dV}{dt}\right)^2}=\frac{\frac{d^2A}{dt^2}-\frac{dA}{dV}\frac{d^2V}{dt^2}}{\left(\frac{dV}{dt}\right)^2},$$

where $\nu(x)$ is unit normal pointed out at $x\in\Sigma(V_0)$. Finally, by Cauchy inequality $|\mathrm{II}|^2\ge\frac{1}{n-1}H^2$, and $Ric(\nu,\nu)\ge\epsilon\cdot Ric_0$, we get

$$\begin{aligned}
A_{V_0}''(V_0)A_{V_0}(V_0)^2 &= A_{V_0}''(0)-A_{V_0}'(V_0)V''(0)\\
&= \frac{d}{dt}\int_{\Sigma(V_0)} H\,d\mu - H\int_{\Sigma(V_0)} H\,d\mu\\
&= \int_{\Sigma(V_0)} \dot{H}\,d\mu\\
&= \int_{\Sigma(V_0)} -|\mathrm{II}|^2-Ric(\nu,\nu)d\mu.
\end{aligned}$$

Hence, we get

$$A_{V_0}''(V_0)\le-\frac{1}{A_{V_0}(V_0)}\left(\frac{1}{n-1}A_{V_0}'(V_0)^2+\epsilon\cdot Ric_0\right). \tag{5}$$

We say that

$$A''(V) \le -\frac{1}{A(V)}\left(\frac{1}{n-1}A'(V)^2 + \epsilon \cdot Ric_0\right). \tag{6}$$

in the sense of comparison functions, which means that for all $V_0 \ge 0$ there exists a smooth function $A_{V_0}(V) \ge A(V)$ with $A_{V_0}(V_0) = A(V_0)$ satisfying

$$A_{V_0}''(V_0) \le -\frac{1}{A_{V_0}(V_0)}\left(\frac{1}{n-1}A_{V_0}'(V_0)^2 + \epsilon \cdot Ric_0\right).$$

Remark2.3: If $A(V)$ has second derivative at V_0, then we define $f(V) = A_{V_0}(V) - A(V)$. For $f(V)$, we have $f(V_0) = 0, f(V) \ge 0$. So, $f'(V_0) = 0, f''(V_0) \ge 0$, namely $A'(V_0) = A_{V_0}'(V_0), A''(V_0) \le A_{V_0}''(V_0)$. Then we have

$$A''(V_0) \le A_{V_0}''(V_0) \le -\frac{1}{A_{V_0}(V_0)}\left(\frac{1}{n-1}A_{V_0}'(V_0)^2 + \epsilon \cdot Ric_0\right) = -\frac{1}{A(V_0)}\left(\frac{1}{n-1}A'(V_0)^2 + \epsilon \cdot Ric_0\right).$$

So, actually at this time the inequality also holds for $A(V)$.

The discussions above can lead to an interesting fact that the minimizing surface Σ has exactly one component.

Proposition2.4 *Suppose* $\Sigma = \partial R$ *minimizes area for its volume,* $R \subseteq (M^n, g)$ *, and* $Ric(g) \ge \delta > 0$. *Then* Σ *has exactly one component.*

Proof: Assume Σ consists of multiple components. Then we just consider two of them, denoted by Σ_1 and Σ_2. We do a variation on Σ which is flowing out on Σ_1, while is flowing in on Σ_2, such that $V_1(t) + V_2(t) = V_1(0) + V_2(0)$, where the $V_i(t)$ denotes the volume contained in the $\Sigma_i(t)$(we can achieve this variation by controlling the ratio of the flowing rate). Now let's consider variations on Σ_1 and Σ_2 respectively. We denote $A_{V_1(0)}(V)$ for Σ_1, $A_{V_2(0)}(V)$ for Σ_2. It is obviously that Σ_1 and Σ_2 must be surfaces which minimize area. So we get

$$A_{V_1(0)}''(V_1(0)) \le -\frac{1}{A_{V_1(0)}(V_1(0))}\left(\frac{1}{n-1}A_{V_1(0)}'(V_1(0))^2 + \epsilon \cdot Ric_0\right) < 0.$$

Here we take $\epsilon \cdot Ric_0 = \delta$.

Despite the fact that flow direction on Σ_2 is inward, since both inward and outward flow have the same second derivative. So,

$$A_{V_2(0)}''(V_2(0)) \le -\frac{1}{A_{V_2(0)}(V_2(0))}\left(\frac{1}{n-1}A_{V_2(0)}'(V_2(0))^2 + \epsilon \cdot Ric_0\right) < 0.$$

Since all $\Sigma(t)$ contain the same volume, and $\Sigma(0)$ is the minimizer, so we must have $A'(0) = A_1'(0) + A_2'(0) = 0$, and $A''(0) \ge 0$, where $A_i(t)$ is the area of $\Sigma_i(t)$, and $A(t)$ is the area of $\Sigma(t)$. What's more, since the volume is constant, we have $V_1'(0) + V_2'(0) = 0, V_1''(0) + V_2''(0) = 0$. Then, by (4) we have

$$
\begin{aligned}
A''(0) = {} & A_1''(0) + A_2''(0) \\
= {} & A_{V_1(0)}''(V_1(0))A_{V_1(0)}(V_1(0))^2 + A_{V_1(0)}'(V_1(0))V_1''(0) \\
+ {} & A_{V_2(0)}''(V_2(0))A_{V_2(0)}(V_2(0))^2 + A_{V_2(0)}'(V_2(0))V_2''(0) \\
< {} & A_{V_1(0)}'(V_1(0))V_1''(0) + A_{V_2(0)}'(V_2(0))V_2''(0) \\
= {} & A_1'(0)(V_1'(0))^{-1}V_1''(0) + A_2'(0)(V_2'(0))^{-1}V_2''(0) \\
= {} & 0.
\end{aligned}
$$

Contradiction.

Remark: Even though in this case the flow rate of the variation is not constant, we are still permitted to apply (5) since it only deals with a derivative with respect to volume.

4. Bray's Proof of Bishop's Theorem

We can start proving the special case of Bishop's theorem with the help of the above preparations. First, we establish a lemma that is in line with common sense.

Lemma3.1 *Suppose* (M^n, g) *satisfies* $Ric(g) \geq \delta > 0$. *Then* $A(V)$ *is strictly increasing on the interval* $\left[0, \frac{1}{2}\mathrm{Vol}(M^n)\right]$.

Proof: The boundaries of a region and its complement are identical, so the statement $A(V) = A\left(\mathrm{Vol}(M^n) - V\right)$ is true in all cases. According to remark 2.3, if $A(V)$ has second derivative everywhere, then the inequality (6) holds for $A(V)$(where we take $\epsilon \cdot Ric_0 = \delta$),which means that $A''(V)$ is strictly negative. The lemma then follows. If $A(V)$ doesn't have second derivative, inequality (6) still applies because the existence of such comparison functions will force $A(V)$ to be strictly concave, which means that $A(V)$ is strictly increasing.

We define

$F(V) = A(V)^{\frac{n}{n-1}}$

and opt to deal with $F(V)$ rather than $A(V)$. The inequality for $F(V)$ ends up being simpler than the inequality for $A(V)$ because $F(V)$ and V have the same units and $F(V)$ is roughly a linear function of V for small V.

By considering $F(V)$, (6) becomes

$$F''(V) \leq -\frac{n}{n-1} F(V)^{-\frac{n-2}{n}} \epsilon \cdot Ric_0 \tag{7}$$

in the sense of comparison functions.

Since $F(V)$ is monotone increasing on [0, $\frac{1}{2}$ Vol(M^n)] and monotone decreasing on [$\frac{1}{2}$ Vol(M^n),Vol(M^n)], $F'(V)$ exists almost everywhere but $F'(V)$ does not necessarily exist for all V, although it is not required to exist for all V. However, the left and right hand derivatives, $F'_-(V)$ and $F'_+(V)$, always exist.

Remark: There are two main reasons for this. First, when V is small, we have $F(V) \approx n(\omega_{n-1})^{\frac{1}{n-1}} V$, where ω_{n-1} is the volume of the sphere S^{n-1}. This results from the fact that the surface $\Sigma(V)$ that minimizes area would fall in one normal coordinate $\exp_p(U)$ and be relatively close to the point p when V is small. Since $\exp_p$ is almost an isometry near $0 \in T_pM$, the surface $\Sigma(V)$ that minimizes area is almost a sphere, just like in the case of Euclidean space. Direct calculation has given us

$$V_n = \begin{cases} \dfrac{R^{2m}}{m!}\pi^m, \ n = 2m \\ \dfrac{2^{m+1}R^{2m+1}}{(2m+1)!!}\pi^m, n = 2m+1 \end{cases} \qquad S_n = \begin{cases} \dfrac{2R^{2m-1}}{(m-1)!}\pi^m, n = 2m \\ \dfrac{2^{m+1}R^{2m}}{(2m-1)!!}\pi^m, n = 2m+1 \end{cases}$$

where V_n is the volume of n dimentional ball with radius R, and S_n the volume of $n-1$ dimentional sphere with radius R. Then ,we have

$$\frac{(S_n)^{\frac{n}{n-1}}}{V_n} = n(\omega_{n-1})^{\frac{1}{n-1}}$$

We therefore know that $\dfrac{F(V+\delta)-F(V)}{\delta}$ must be bounded for small V and small δ. The same argument applies to V near $\mathrm{Vol}(M^n)$. So for V near zero and $\mathrm{Vol}(M^n)$, we know that $F_+'(V)$ and $F_-'(V)$ will always exist.

Second, although one side derivative of $F(V)$ may not exist, inequality (7), which holds in the context of comparison functions, implies

$$\lim_{\delta\to 0^+}\frac{F(V+\delta)-F(V)}{\delta} \text{ and } \lim_{\delta\to 0^-}\frac{F(V+\delta)-F(V)}{\delta}$$

are decreasing. However, based on the discussion above, we can conclude $F_+'(V)$ and $F_-'(V)$ will always exist for V near zero and $\mathrm{Vol}(M^n)$. This implies the above two limits exist for any $V \in (0, \mathrm{Vol}(M^n))$ since they are bounded below and above. Namely $F_+'(V)$ and $F_-'(V)$ will always exist. Furthermore it is evident that $F_+'(V) \le F_-'(V)$ for all V.

Given inequality (7), it is natural to want to integrate it. We define the Ricci curvature mass for $V < \mathrm{Vol}(M^n)$:

$$m(V) = \left(n^2(\omega_{n-1})^{\frac{2}{n-1}} - F_+'(V)^2\right) - \frac{n^2 Ric_0}{n-1}F(V)^{\frac{2}{n}},$$

and define $m(\mathrm{Vol}(M^n)) = 0$. It is evident that $m(0) = 0$.

Lemma3.2 *Under the same assumption as theorem 1.2,* $m(V)$ *of* M^n *is non-decreasing on the internal* $\left[0, \frac{1}{2}\mathrm{Vol}(M^n)\right]$

proof: The main idea is that if $F(V)$ were smooth, then we have

$$m'(V) = -2F'(V)\left(F''(V) + \frac{n}{n-1}Ric_0 F(V)^{-\frac{n-2}{n}}\right).$$

Since $F(V)$ is increasing on $\left[0, \frac{1}{2}\mathrm{Vol}(M^n)\right]$, $F'(V) \ge 0$, and by remark 2.3 and inequality (7), we know that $m'(V) \ge 0$ on $\left[0, \frac{1}{2}\mathrm{Vol}(M^n)\right]$, i.e. $m(V)$ is non-decreasing on $\left[0, \frac{1}{2}\mathrm{Vol}(M^n)\right]$.

More generally, it is sufficient to prove that $m'(V) \ge 0$ distributionally. We define

$$\tilde{F}(V)=\begin{cases}0 & V\le 0\\ F(V) & 0<V<\frac{1}{2}\mathrm{Vol}(M^n)\\ F\left(\frac{1}{2}\mathrm{Vol}(M^n)\right) & V\ge\frac{1}{2}\mathrm{Vol}(M^n)\end{cases}$$

and define $\tilde{m}(V)$ with respect to $\tilde{F}(V)$.

To prove that $m'(V)\ge 0$ distributionally, using the integration by parts formula, it is sufficient to prove

$$-\int_{-\infty}^{+\infty}\tilde{m}(V)\phi'(V)dV\ge 0$$

for all smooth positive test functions ϕ with compact support in $\left[0,\frac{1}{2}\mathrm{Vol}(M^n)\right]$. The finite difference operator Δ_δ is needed for the following proof, which is defined as

$$\Delta_\delta\left(g(V)\right)=\frac{g(V+\delta)-g(V)}{\delta}$$

Then,

$$\begin{aligned}-\int_{-\infty}^{+\infty}\tilde{m}(V)\phi'(V)dV &= -\int_{-\infty}^{+\infty}\left(\left(n^2(\omega_{n-1})^{\frac{2}{n-1}}-\tilde{F}_+'(V)^2\right)-\frac{n^2Ric_0}{n-1}\tilde{F}(V)^{\frac{2}{n}}\right)\phi'(V)dV\\ &= -\lim_{\delta\to 0^+}\int_{-\infty}^{+\infty}\{\left(n^2(\omega_{n-1})^{\frac{2}{n-1}}-\Delta_\delta\left(\tilde{F}(V)\right)^2\right)-\frac{n^2Ric_0}{n-1}\tilde{F}(V)^{\frac{2}{n}}\}\Delta_\delta\left(\phi(V)\right)dV\\ &= \lim_{\delta\to 0^+}\int_{-\infty}^{+\infty}\Delta_{-\delta}\{\left(n^2(\omega_{n-1})^{\frac{2}{n-1}}-\Delta_\delta\left(\tilde{F}(V)\right)^2\right)-\frac{n^2Ric_0}{n-1}\tilde{F}(V)^{\frac{2}{n}}\}\phi(V)dV\end{aligned}$$

where we have used $\int f(x)\Delta_\delta\left(g(x)\right)dx=-\int\Delta_{-\delta}\left(f(x)\right)g(x)dx$ .Then since $F(V)$ has left-hand derivatives everywhere, in the limit, we have

$$=-\lim_{\delta\to 0^+}\int_{-\infty}^{+\infty}\left(\Delta_{-\delta}\left(\Delta_\delta\left(\tilde{F}(V)\right)^2\right)+\frac{2n}{n-1}Ric_0\tilde{F}(V)^{-\frac{n-2}{n}}\tilde{F}_-'(V)\right)\phi(V)dV$$

By using the comparison functions at each point, we can deduce from remark 2.3 that if $F''(V)$ exists at V_0 , $F''(V_0)\le F_{V_0}''(V_0)$. So, in this case, as $F(V_0)=F_{V_0}'(V_0)$, we have $F(V_0)F''(V_0)\le F_{V_0}'(V_0)F_{V_0}''(V_0)$. Then since $\lim_{\delta\to 0^+}\Delta_{-\delta}\left[\Delta_\delta\left(\tilde{F}(V)\right)^2\right]|_{V=V_0}=2F(V_0)F''(V_0)$, we have

$$\lim_{\delta\to 0^+}\Delta_{-\delta}\left[\Delta_\delta\left(\tilde{F}(V)\right)^2\right]|_{V=V_0}\le\lim_{\delta\to 0^+}\Delta_{-\delta}\left[\Delta_\delta\left(F_{V_0}(V)\right)^2\right]|_{V=V_0}.$$

The inequality still holds even if $F(V)$ does not have the second derivative at V_0, because, similar to lemma 3.1, $F(V)$ is under a concave function $F_{V_0}(V)$, satisfying $F_{V_0}(V)\ge F(V)$, $F_+'(V)\ge 0$, and $F_+'(V)$ is decreasing. If you draw a picture of the situation, the conclusion is clear.

Changing the integration variable to V_0, then, we have

$$-\int_{-\infty}^{+\infty}\tilde{m}(V)\phi'(V)dV\ge-\lim_{\delta\to 0^+}\int_{-\infty}^{+\infty}\left(\Delta_{-\delta}\left(\Delta_\delta\left(F_{V_0}(V_0)\right)^2\right)+\frac{2n}{n-1}Ric_0\tilde{F}(V_0)^{-\frac{n-2}{n}}\tilde{F}_-'(V_0)\right)\phi(V_0)dV_0,$$

since $\tilde{F}_+'(V_0)=\tilde{F}_-'(V_0)=\tilde{F}'(V_0)=F'(V_0)=F_{V_0}'(V_0)$ except at a countable number of points, further we have

$$= \quad -\lim_{\delta\to 0^+}\int_{-\infty}^{+\infty}\left(\Delta_{-\delta}\left(\Delta_{\delta}\left(F_{V_0}(V_0)\right)^2\right)+\frac{2n}{n-1}Ric_0F(V_0)^{-\frac{n-2}{n}}F_{V_0}'(V_0)\right)\phi(V_0)dV_0$$
$$= -\lim_{\delta\to 0^+}\int_{-\infty}^{+\infty}2\,F_{V_0}'(V_0)F_{V_0}''(V_0)+\frac{2n}{n-1}Ric_0F(V_0)^{-\frac{n-2}{n}}F_{V_0}'(V_0))\phi(V_0)dV_0$$
$$= -\lim_{\delta\to 0^+}\int_{-\infty}^{+\infty}2\,F_{V_0}'(V_0)\{F_{V_0}''(V_0)+\frac{n}{n-1}Ric_0F(V_0)^{-\frac{n-2}{n}}\}\phi(V_0)dV_0$$
$$\geq 0.$$

The last inequality follows from (7). Hence, $m'(V)\geq 0$ distributionally, so $m(V)$ is a nondecreasing function of V.

proof of Theorem 1.2: Review some concepts we just mentioned.

$m(V)=\left(n^2\left(\omega_{n-1}\right)^{\frac{2}{n-1}}-F_+'(V)^2\right)-\frac{n^2Ric_0}{n-1}F(V)^{\frac{2}{n}}$.

And,

$$F''(V)\leq -\frac{n}{n-1}F(V)^{-\frac{n-2}{n}}\cdot Ric_0 \tag{8}$$

in the sense of comparison functions, which means that for all $V_0\geq 0$ there exists a smooth function $F_{V_0}(V)\geq F(V)$ with $F_{V_0}(V_0)=F(V_0)$ satisfying

$F_{V_0}''(V_0)\leq -\frac{n}{n-1}F_{V_0}(V_0)^{-\frac{n-2}{n}}\cdot Ric_0$

Now consider phase space which we will view as the $x-y$ plane where $x=F(V)$ and $y=F_+'(V)$. Let γ be the path in phase space of F(V) for V between 0 and $\frac{1}{2}\mathrm{Vol}(M^n)$. Since $F'(V)$ exists almost everywhere, and $F_+'(V)=F'(V)$ holds for these poins, then we note that

$$\frac{1}{2}\mathrm{Vol}(M^n)=\int_\gamma dV=\int_\gamma\frac{dx}{y}. \tag{9}$$

From the above discussion we know that $F(0)=0, F_+'(V)\geq 0$ for $V\in\left(0,\frac{1}{2}\mathrm{Vol}(M^n)\right)$. Since on $\left(0,\frac{1}{2}\mathrm{Vol}(M^n)\right)$ $F(V)$ is strictly increasing and $F_+'(V)$ is strictly decreasing ($F(V)$ is strictly concave), the x position of γ is non-decreasing and the y position of γ is strictly decreasing. Since $F_+'(V)$ may not be continuous, γ may be disconnected. Because $F_+'\left(\frac{1}{2}\mathrm{Vol}(M^n)\right)\leq 0$, we set $F_+'\left(\frac{1}{2}\mathrm{Vol}(M^n)\right)=0$. Besides, since this action won't have an impact on the integration, we can use vertical lines to connect the jumping points. We still denote the new curve as γ.

Now we want to find the γ which maximizes equation (9), with the constraint inequality (8) in the sense of comparison functions. If $F(V)$ has second derivative, since $F''(V)=y\frac{dy}{dx}$, we can rewrite inequality (8) as

$$y\frac{dy}{dx}\le -\frac{n}{n-1}x^{-\frac{n-2}{n}}Ric_0 \Leftrightarrow \frac{dx}{y}\le \frac{dy}{-\frac{n}{n-1}x^{-\frac{n-2}{n}}Ric_0}.$$

where the latter is true in the forward direction ($dx\ge 0$).

Because of the existence of comparison functions, the same estimation still works when $F''(V)$ does not exist. From the above inequality, since the total dy is fixed, namely $n(\omega_{n-1})^{\frac{1}{n-1}}$, we know that the curve which maximizes equation (9) must have equality in (8) and have the first coordinate x of the terminal point $(x_0,0)$ as big as possible. As we have computed in lemma 3.2, the equality in (8) is equivalent to $m'(V)=0$, which means that $m(V)\equiv 0$. Additionally, since we have

$$m(V)=\left(n^2(\omega_{n-1})^{\frac{2}{n-1}}-F_+'(V)^2\right)-\frac{n^2 Ric_0}{n-1}F(V)^{\frac{2}{n}},$$

the terminal point $(x_0,0)$ satisfies

$$m\left(\frac{1}{2}\mathrm{Vol}(M^n)\right)=n^2(\omega_{n-1})^{\frac{2}{n-1}}-\frac{n^2 Ric_0}{n-1}x_0^{\frac{2}{n}}.$$

Since $m(V)$ is non-decreasing on $\left[0,\frac{1}{2}\mathrm{Vol}(M^n)\right]$, $m(V)\equiv 0$ is the case maximizes x_0. But the standard sphere (S^n,g_0) with constant Ricci curvature $Ric_0\cdot g_0$ has equality in (8)(this can be seen in the deduction of the inequality for $A''(V)$). Let γ_0 be the path in phase space corresponding to this standard sphere with zero mass. Then

$$\frac{1}{2}\mathrm{Vol}(M^n)=\int_\gamma\frac{dx}{y}\le\sup_\gamma\int_\gamma\frac{dx}{y}=\int_{\gamma_0}\frac{dx}{y}=\frac{1}{2}\mathrm{Vol}(S^n),$$

which completes the proof.

Remark: Although Bishop's theorem's isoperimetric proof appears to be cumbersome, the real difficulty arises when the second derivative is nonexistent. The main challenge is that $A(V)$ might not be a smooth function. The proof can be considerably simplified if we only take into account the smooth case. It's also interesting to note that using the same method, as described in [8-11], one can estimate volume given a constraint scalar curvature condition.

5. Afterwards

The idea of transforming a geometric problem into an analytic problem is inspiring. Under the guidance of this idea, we consider the geodesic ball $B_r(p)$ and its boundary $S_r(p)$. Under the same assumption as theorem 1.2, we define $A(t)=|S_t(p)|$, where $|\cdot|$ means the corresponding volume. This time $A(t)$ is smooth. The same computation in section 2 follows, we can get

$$A'(t)=\int_{S_t(p)}Hd\mu,\ A''(t)=\int_{S_t(p)}\left(-|\mathrm{II}|^2-Ric(\nu,\nu)+H^2\right)d\mu$$

where $\nu(x)$ is unit normal pointed out at $x\in\Sigma(V_0)$. By Cauchy inequality $|\mathrm{II}|^2\ge\frac{1}{n-1}H^2$, and $Ric(\nu,\nu)\ge\epsilon\cdot Ric_0$, we have

$$A''(t)\le -Ric_0A(t)+\frac{n-1}{n}\int_{S_t(p)}H^2d\mu.$$

However, in this case we fail to associate $\frac{n-1}{n}\int_{S_t(p)} H^2 d\mu$ with $A(t)$ and $A'(t)$, because there is no conclusion that the mean curvature of $S_t(p)$ is constant. If the mean curvature of $S_t(p)$ is constant, then $S_t(p)$ must possess some degree of symmetry. For arbitrary points on manifolds, this is not always the case. And we cannot use the Cauchy-Schwarz inequality, for

$$A(t)\int_{S_t(p)} H^2 d\mu \geq \left(\int_{S_t(p)} H d\mu\right)^2 = A'(t)^2.$$

If there are some comparison theorems about mean curvature, then it will be possible to use the same method to get an ordinary differential inequality to prove this theorem.

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